

The Helical Wiggler

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(Oct. 12, 1986)

1 Problem

A variant on the electro- or magnetostatic boundary value problem arises in accelerator physics, where a specified field, say $\mathbf{B}(0, 0, z)$, is desired along the z axis. In general there exist static fields $\mathbf{B}(x, y, z)$ that reduce to the desired field on the axis, but the “boundary condition” $\mathbf{B}(0, 0, z)$ is not sufficient to insure a unique solution.

For example, find a field $\mathbf{B}(x, y, z)$ that reduces to

$$\mathbf{B}(0, 0, z) = B_0 \cos kz \hat{\mathbf{x}} + B_0 \sin kz \hat{\mathbf{y}} \quad (1)$$

on the z axis. In this, the magnetic field rotates around the z axis as z advances.

The use of rectangular or cylindrical coordinates leads “naturally” to different forms for \mathbf{B} . One 3-dimensional field extension of (1) is the so-called helical wiggler [1], which obeys the auxiliary requirement that the field at $z + \delta$ be the same as the field at z , but rotated by angle $k\delta$.

2 Solution

2.1 Solution in Rectangular Coordinates

We first seek a solution in rectangular coordinates, and expect that separation of variables will apply. Thus, we consider the form

$$B_x = f(x)g(y) \cos kz, \quad (2)$$

$$B_y = F(x)G(y) \sin kz, \quad (3)$$

$$B_z = A(x)B(y)C(z). \quad (4)$$

Then

$$\nabla \cdot \mathbf{B} = 0 = f'g \cos kz + FG' \sin kz + ABC', \quad (5)$$

where the $'$ indicates differentiation of a function with respect to its argument. Equation (5) can be integrated to give

$$ABC = -\frac{f'g}{k} \sin kz + \frac{FG'}{k} \cos kz. \quad (6)$$

The z component of $\nabla \times \mathbf{B} = 0$ tells us that

$$\frac{\partial B_x}{\partial y} = fg' \cos kz = \frac{\partial B_y}{\partial x} = F'G \sin kz, \quad (7)$$

which implies that g and F are constant, say 1. Likewise,

$$\frac{\partial B_x}{\partial z} = -fk \sin kz = \frac{\partial B_z}{\partial x} = A'BC = -\frac{f''}{k} \sin kz, \quad (8)$$

using (6-7). Thus, $f'' - k^2 f = 0$, so

$$f = f_1 e^{kx} + f_2 e^{-kx}. \quad (9)$$

Finally,

$$\frac{\partial B_y}{\partial z} = Gk \cos kz = \frac{\partial B_z}{\partial y} = AB'C = \frac{G''}{k} \sin kz, \quad (10)$$

so

$$G = G_1 e^{ky} + G_2 e^{-ky}. \quad (11)$$

The “boundary conditions” $f(0) = B_0 = G(0)$ are satisfied by

$$f = B_0 \cosh kx, \quad G = B_0 \cosh ky, \quad (12)$$

which together with (6) leads to the solution

$$B_x = B_0 \cosh kx \cos kz, \quad (13)$$

$$B_y = B_0 \cosh ky \sin kz, \quad (14)$$

$$B_z = -B_0 \sinh kx \sin kz + B_0 \sinh ky \cos kz, \quad (15)$$

This satisfies the last “boundary condition” that $B_z(0, 0, z) = 0$.

However, this solution does not have helical symmetry.

2.2 Solution in Cylindrical Coordinates

Suppose instead, we look for a solution in cylindrical coordinates (r, θ, z) . We again expect separation of variables, but we seek to enforce the helical symmetry that the field at $z + \delta$ be the same as the field at z , but rotated by angle $k\delta$. This symmetry implies that the argument kz should be replaced by $kz - \theta$, and that the field has no other θ dependence.

We begin constructing our solution with the hypothesis that

$$B_r = F(r) \cos(kz - \theta), \quad (16)$$

$$B_\theta = G(r) \sin(kz - \theta). \quad (17)$$

To satisfy the condition (1) on the z axis, we first transform this to rectangular components,

$$B_z = F(r) \cos(kz - \theta) \cos \theta + G(r) \sin(kz - \theta) \sin \theta, \quad (18)$$

$$B_y = -F(r) \cos(kz - \theta) \sin \theta + G(r) \sin(kz - \theta) \cos \theta, \quad (19)$$

from which we learn that the “boundary conditions” on F and G are

$$F(0) = G(0) = B_0. \quad (20)$$

A suitable form for B_z can be obtained from $(\nabla \times \mathbf{B})_r = 0$:

$$\frac{1}{r} \frac{\partial B_z}{\partial \theta} = \frac{\partial B_\theta}{\partial z} = kG \cos(kz - \theta), \quad (21)$$

so

$$B_z = -krG \sin(kz - \theta), \quad (22)$$

which vanishes on the z axis as desired.

From either $(\nabla \times \mathbf{B})_\theta = 0$ or $(\nabla \times \mathbf{B})_z = 0$ we find that

$$F = \frac{d(rG)}{dr}. \quad (23)$$

Then, $\nabla \cdot \mathbf{B} = 0$ leads to

$$(kr)^2 \frac{d^2(krG)}{d(kr)^2} + kr \frac{d(krG)}{d(kr)} - [1 + (kr)^2](krG) = 0. \quad (24)$$

This is the differential equation for the modified Bessel function of order 1 [2]. Hence,

$$G = C \frac{I_1(kr)}{kr} = \frac{C}{2} \left[1 + \frac{(kr)^2}{8} + \dots \right], \quad (25)$$

$$F = C \frac{dI_1}{d(kr)} = C \left(I_0 - \frac{I_1}{kr} \right) = \frac{C}{2} \left[1 + \frac{3(kr)^2}{8} + \dots \right]. \quad (26)$$

The “boundary conditions” (20) require that $C = 2B_0$, so our second solution is

$$B_r = 2B_0 \left(I_0(kr) - \frac{I_1(kr)}{kr} \right) \cos(kz - \theta), \quad (27)$$

$$B_\theta = 2B_0 \frac{I_1}{kr} \sin(kz - \theta), \quad (28)$$

$$B_z = -2B_0 I_1 \sin(kz - \theta), \quad (29)$$

which is the form discussed in [1].

3 References

- [1] J.P. Blewett and R. Chasman, *Orbits and fields in the helical wiggler*, J. Appl. Phys. **48**, 2692-2698 (1977).
- [2] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D.C., 1964), sec. 9.6